

ON AVERAGING OF PERIODIC SYSTEMS

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Arguments are presented which permit the determination at the "physical" level of rigor of the asymptotic nature of problem solutions for bodies with a periodic microstructure. A simple method is given of constructing the averaged equations. The exposition is carried out on two examples: the problem of the deformation of an elastic body with periodically distributed cavities and the problem of the flow of a viscous liquid past periodically distributed particles. In various areas of the mechanics of continuous media the problem arises of the averaged description of bodies consisting of a large number of periodically recurring elements (cells). The case when the state of the cell is given by a finite collection of parameters (for example, an atomic lattice in the harmonic approximation or a rod system) was studied in [1]. The case when the cell's state is described by a collection of field functions which satisfy a system of partial differential equations was investigated in [2-12]. The averaging of one second-order linear elliptic equation was considered in [2]. The asymptotics and the averaged equations for a wide class of linear equation systems were constructed in [3, 4] independently of [2]. Averaged equations in variational problems were derived in [5]. The results obtained in [3, 4] were extended to nonlinear equations in [5, 6]. Further mathematical proof and generalizations to variational inequalities are contained in [7-11]. The results in [5] were recently obtained independently in [12]. The averaged equations in [2-12] were obtained under the following restrictions: if there is a cavity in the cell, then Neumann-type conditions are imposed on its boundary. The corresponding class of problems contains, in particular, the problems on the efficiency coefficients of heat- and electro-conductivity of periodically micro-inhomogeneous bodies, on the potential flow of an ideal incompressible liquid past periodically situated obstacles, on the determination of the stressed state of an elastic body with periodically situated cracks, etc. However, it does not include, for instance, the problem on the flow of a viscous liquid through the periodic lattice of bodies. This is connected with the fact that the transition from Neumann-type conditions to Dirichlet-type conditions essentially alters the asymptotic behavior of the solution. It is shown below that the original linear problem can be rewritten in other terms such that the question on the construction of the solution's asymptotics and of the averaged equations is simple to resolve both for a Neumann-type boundary condition as well as for a Dirichlet-type boundary condition. The idea of a quasicontinuum [1] is used in the reformulation. The averaged equations obtained in the viscous liquid problem, as assumed earlier in [13-16], contain an asymmetric tensor of viscous stresses. Certain generalizations are discussed.

1. Statement of the problem. In a three-dimensional space R we consider a periodic lattice, cubic for simplicity, with a step b . In each cell we locate a certain region A such that this region recurs periodically in the space. By B_n we denote the cell numbered n (n is an integer-valued vector with components n^i) and by A_n we denote the region A in cell B_n . We introduce a certain bounded region V with diameter l , $l \gg b$. In the region $V - \Sigma A_n$ (the A_n wholly belonging to V occur in the sum) we consider the system of equations

$$-\partial p / \partial x^i + \mu \Delta w_i = 0 \tag{1.1}$$

$$\partial w^i / \partial x^i = 0 \tag{1.2}$$

On the boundary ∂V of region V we prescribe the functions $w_i(x)$

$$w_i = h_i(x) \quad \text{on } \partial V \tag{1.3}$$

On the boundaries ∂A_n of regions A_n we consider two types of boundary conditions

$$(-p \delta_{ik} + 2\mu \varepsilon_{ik}) v^k = 0 \quad \text{on } \partial A_n \quad (\text{Problem 1}) \tag{1.4}$$

$$\varepsilon_{ik} = w_{(i,k)} \equiv 1/2 (w_{i,k} + w_{k,i})$$

$$w_i = u_i(n) + e_{ijk} \omega^j(n) (x^k - b n^k) \quad \text{on } \partial A_n \quad (\text{Problem 2}) \tag{1.5}$$

Here v^k are the components of the unit normal vector, $u_i(n)$ and $\omega_i(n)$ are vectors specified in each cell, δ_{ij} and e_{ijk} are the Kronecker and Levi-Civita symbols, differentiation with respect to the cartesian coordinates x^i is denoted by a comma in the indices, the symmetrization operation is denoted by parentheses.

Problem 1 is the problem on the deformation of an incompressible isotropic elastic body containing a large number of traction-free periodically situated cavities A_n ; w_i are the components of the displacement vector, p is the pressure, μ is the shear modulus. Problem 2 is the problem on the flow of a viscous incompressible liquid (in the Stokes approximation) past a large number of periodically situated particles A_n having a translational velocity $u_i(n)$ and an angular velocity $\omega_i(n)$; w_i are the components of the liquid's velocity vector, p is the pressure, μ is the shear viscosity coefficient. In the cell, Problem 1 is in the nature of a Neumann problem and Problem 2, of a Dirichlet problem. The assumptions on the incompressibility and isotropy of the elastic body, connected with Problem 1, are unessential and are adopted in order to treat within the framework of one set of equations the physically interesting problems of Dirichlet and Neumann types.

Problems 1 and 2 can be given variational formulations. Problem 1 can be treated as a problem on the minimum of the functional

$$E = 2\mu \int_{V - \Sigma A_n} \varepsilon_{ij} \varepsilon^{ij} d^3x \tag{1.6}$$

under constraints (1.2) and (1.3) and Problem 2 as a problem on the minimum of functional (1.6) under constraints (1.2), (1.3) and (1.5). The pressure p coincides with the Lagrange multiplier for constraint (1.2).

If $b/l \ll 1$, then region V contains a large number of cavities A_n of order $(b/l)^3$, and an averaged description becomes possible. We are required to construct the corresponding averaged equations. The averaged equations are in some sense limit

equations as $b/l \rightarrow 0$. In the limit passage $b/l \rightarrow 0$ we assume the region V to be fixed. This means that we are examining a sequence of periodic lattices with a decreasing step b . The volume concentration c , the ratio of the volume of region A_n to the volume of an elementary cell, remains finite in general. The behavior of the functions $u_i(n)$, $\omega_i(n)$ and $h_i(x)$ as $b \rightarrow 0$ is described below.

2. On the concept of averages. The solutions of Problems 1 and 2 are rapidly-oscillating functions of the coordinates. Usually we are interested in certain average quantities changing little on distances of the order of b . We define the average characteristics in the following manner. A function $f(x)$ continuous in the closed region V is called a macrofunction if it is independent of parameter b . The naturalness of such a definition is connected with the fact that for each function $f(x)$ continuous in a closed region we can find b so small that everywhere in region V function $f(x)$ varies little on distances of the order of b .

Let function $F(x, b)$ be specified in region $V - \Sigma A_n$. The domain of determination of $F(x, b)$, as the function itself, depends upon b . If for $b \rightarrow 0$ function $F(x, b)$ is representable in the form

$$F(x, b) = f(x) + r(x, b) \quad (2.1)$$

where function $f(x)$ is defined and continuous in the closed region V and is independent of b , while $r(x, b) \rightarrow 0$ as $b \rightarrow 0$, then function $f(x)$ is called the asymptotic average of function $F(x, b)$. Analogously, if a function $G(n^i, b)$ of the integer arguments n^i is representable in the form

$$G(n^i, b) = g(n^i) + r(n^i, b)$$

where $g(n^i)$ is a macrofunction and $r(n^i, b) \rightarrow 0$ as $b \rightarrow 0$, then $g(n^i)$ is called the asymptotic average of function $G(n^i, b)$.

From function $F(x, b)$ we can construct a function $G(n, b)$ of a discrete argument by the rule (τ is the volume of region $B_n - A_n$)

$$G(n, b) = \frac{1}{\tau} \int_{B_n - A_n} F(x, b) d^3x \quad (2.2)$$

If the function $G(n, b)$ of (2.2) has the asymptotic average $g(x)$, then $g(x)$ is called the volume-average of $F(x, b)$. If function $F(x, b)$ has the asymptotic average $f(x)$, then it obviously has a volume-average and the asymptotic average coincides with the volume-average. The converse is not true: an asymptotic average may not exist for a function $F(x, b)$ having a volume-average. For example, the function $\sin^2(x/b)$ on the straight line $f(x)$ does not have an asymptotic average, however, a volume-average exists for it. From now on we shall omit the argument b for brevity when speaking of functions of the type of $F(x, b)$ and $G(n, b)$, since the dependency on b usually follows from the context.

We shall assume that the functions $u_i(n)$ and $\omega_i(n)$ in boundary conditions (1.5) have the asymptotic averages $u_i(x)$ and $\omega_i(x)$ and that $h_i(x)$ is a macrofunction. The subsequent presentation is based on the following two assumptions.

Assumption 1. Functions $w_i(x)$, being solutions of Problems 1 and 2, have the volume-average $v_i(x)$.

Assumption 2. The asymptotic behavior of functions w_i as $b \rightarrow 0$ is

completely determined by prescribing $v_i(x)$ and is independent of the geometry of region V and of the nature of the boundary conditions on ∂V . (We keep in mind the asymptotic behavior inside region V without taking the boundary effects into account).

For a wide class of problems of type 1 these assumptions can be proved by using the results in [2-12]. Before we present the method for constructing the solution we describe the results.

3. Asymptotics of the solution and averaged equations.

The first terms of the asymptotics of the solution of Problem 1 as $b \rightarrow 0$ are

$$w_i = v_i(x) + b\psi_i(\xi, x) \tag{3.1}$$

Here $v_i(x)$ is a macrofunction, ξ is a vector with components $\xi^k, \xi^k = x^k/b$. The symbol \approx signifies that the terms of an order of smallness higher in comparison with that of the ones written out have been omitted. The functions ψ_i are periodic in ξ^k with period 1. It is convenient to introduce a standard elementary cell—a region B of changing variables ξ^k in the limits $-1/2 \leq \xi^k \leq +1/2$ and to assume that $\xi^k \in B$. In the elementary cell the region A with boundary ∂A corresponds to regions A_n .

Both terms of asymptotic expansion (3.1) are essential for the computation of the stresses, since the derivatives of the displacements w_i in the first approximation are given by the formula

$$\frac{\partial w_i}{\partial x^k} = \frac{\partial v_i}{\partial x^k} + \frac{\partial \psi_i}{\partial \xi^k}$$

Later on, derivatives with respect to ξ^k are denoted by a vertical bar in the indices: $\partial \psi_i / \partial \xi^k \equiv \psi_{i|k}$. The averaged equations serve to determine the average displacements v_i and have the form

$$\frac{\partial}{\partial x^k} \frac{\partial U}{\partial v_{i,k}} = 0 \quad \text{in } V, \quad v_i = h_i \quad \text{on } \partial V \tag{3.2}$$

Here U is a quadratic form in $v_{i,k}$, being the minimal value of a functional

$$U = \inf_{B \rightarrow A} 2\mu \int_{B \rightarrow A} (v_{(i,j)} + \psi_{(i|j)})(v^{(i,j)} + \psi^{(i|j)}) d^3\xi \tag{3.3}$$

The minimum in (3.3) is sought over all periodic functions ψ_i satisfying the incompressibility condition

$$\partial \psi^i / \partial \xi^i = 0 \tag{3.4}$$

The quantities $v_{(i,j)}$ are assumed to be constants. The minimizing functions ψ_i in the variational problem (3.3) depends linearly on $v_{(i,j)}$ and, through $v_{(i,j)}$, depends on x . These functions appear in expansion (3.1). The averaged equations (3.2) are completely defined after the problem is solved on cell (3.3) and the quadratic form U is computed. To solve the problem on the cell we can use the methods developed in [17, 18].

The averaged equations have the form of equilibrium equations of a homogeneous elastic body with elastic energy density U and with stress tensor $\sigma^{ij} = \partial U / \partial v_{i,j}$. We note that because of the presence of cavities the property of incompressibility of the elastic body vanishes and from the macroscopic point of view the elastic body is compressible.

In the consideration of Problem 2 for simplicity we assume that $\omega_i = O(1)$ as $b \rightarrow 0$. Then the second summand in boundary condition (1.5) is considerably less than the first and does not enter into the principal approximation. The first terms of the asymptotic expansion in Problem 2 are

$$w_i = v_i(x) + (u_k(x) - v_k(x)) \varphi_i^k(\xi) + b\psi_i(\xi, x) \tag{3.5}$$

where v_i is a macrofunction and φ_i^k are periodic functions of ξ , which satisfy the conditions

$$\langle \varphi_i^k \rangle = 0, \quad \varphi_i^k|_{\partial A} = \delta_i^k, \quad \frac{\partial \varphi_i^k}{\partial \xi^i} = 0 \tag{3.6}$$

Here $\langle \cdot \rangle$ is the integral with respect to ξ over region $B - A$, referred to $1 - c$. The last term in (3.5) makes a small contribution to the velocity when $v_i \neq 0$ and to the stress when $u_k - v_k \neq 0$.

The averaged equations serve to determine the average velocity $v_i(x)$ and have the form

$$\frac{1}{2} \frac{\partial D}{\partial v_i} - \frac{\partial}{\partial x^k} \frac{1}{2} \frac{\partial D}{\partial v_{i,k}} = - (1 - c) \frac{\partial p}{\partial x^i} \text{ in } V \tag{3.7}$$

$$\frac{\partial}{\partial x^i} ((1 - c) v^i + c u^i) = 0 \text{ in } V \tag{3.8}$$

$$v_i = h_i \text{ on } \partial V \tag{3.9}$$

The dissipation D is the sum of quadratic forms D_1 and D_2 and of the bilinear form D_{12} : $D = D_1 + D_{12} + D_2$; D_1 is a quadratic form in $u_i - v_i$, D_2 is a quadratic form in the components of the deformation velocity vector $e_{ij} = v_{(i,j)}$, of the relative angular velocity vector $\Omega_i = \omega_i - 1/2(\text{rot } v)_i$ and of the relative velocity gradient $\Delta_{ij} = u_{i,j} - v_{i,j}$, and D_{12} contains the cross terms between $u_i - v_i$ and e_{ij} , Ω_i , Δ_{ij} . Quadratic form D_1 is the minimal value of a functional

$$D_1 = \inf 2\mu b^{-2} \int_{B-A} \varphi_{(i|j)} \varphi^{(i|j)} d^3\xi = D^{ij} (u_i - v_i) (u_j - v_j) \tag{3.10}$$

$$\varphi_i = (u_k - v_k) \varphi_i^k$$

The minimum in (3.10) is sought over all periodic functions φ_i^k satisfying conditions (3.6). The vector $u_k - v_k$ in (3.10) is assumed to be constant. Quadratic form

D_2 is the minimal value of a functional

$$D_2 = \inf 2\mu \int_{B-A} \psi_{(i|j)} \psi^{(i|j)} d^3\xi \tag{3.11}$$

The minimum in (3.11) is sought over all functions ψ_i satisfying the constraints

$$\langle \psi_i \rangle = 0, \quad \psi_i|_{\partial A} = e_{ijk} \omega^j \xi^k \tag{3.12}$$

$$\partial \psi^i / \partial \xi^i = 0, \quad [\psi_i]_s = v_{i,s} + \Delta_{ks} \varphi_i^k(\xi)$$

Here $[\psi_i]_s$ is the difference in the values of function ψ_i at the faces $\xi^s = +1/2$ and $\xi^s = -1/2$. The quantities $v_{i,s}$ and Δ_{ks} are assumed to be constant on the cell; in (3.12) φ_i^k is the minimizing function of variational problem (3.10). The bilinear form D_{12} is determined from the solutions of the variational problems (3.10) and (3.11) by the relation

$$D_{12} = 4\mu b^{-1} \int_{B-A} \varphi_{(i|j)} \psi^{(i|j)} d^3\xi \tag{3.13}$$

These same solutions occur in expansion (3.5).

The averaged equations (3.7) are completely determined after the solving of the problems on the cells and the computing of D_1, D_2 and D_{12} . The averaged equations contain four unknown functions: the three velocity components v_i and the pressure p . From (3.10), (3.11) and (3.13) it follows that the coefficients of forms D_1, D_{12} and D_2 are of order b^{-2}, b^{-1} and 1 , respectively. Therefore, D_{12} and D_2 can be discarded in the first approximation. Equations (3.7) take the form of Darcy equations

$$D^{ij} (v_j - u_i) = - (1 - c) \frac{\partial p}{\partial x^i} \tag{3.14}$$

From the three boundary conditions in (3.9) we can retain only one for Eq. (3.14), for example,

$$(v_i - h_i) v^i = 0 \tag{3.15}$$

For $u_i = 0$ the boundary-value problem (3.14), (3.8), (3.15) is the classical problem on the filtering of a liquid through a porous medium. The retaining in Eq. (3.7) of terms connected with forms D_{12} and D_2 enables us to seek a solution satisfying all three boundary conditions in (3.9). At a distance from boundary ∂V this solution will coincide with the solution of the Darcy equations, while close to the boundary it is of the nature of a boundary layer.

We pass on to presenting the method for constructing the solution's asymptotics and the averaged equations.

4. An auxiliary problem. The main idea in the construction of the averaged equations is the following. In the variational Problems 1 and 2 we impose kinematic constraints: the volume-average of functions $w_i(x)$ take specified values $v_i(x)$. We minimize functional E under this additional constraint. The minimal value \bar{E} of functional E is a functional of $v_i(x)$. To find $v_i(x)$ we obviously need to minimize functional \bar{E} over v_i . According to Assumption 2, for the actual computation of \bar{E} we can remove the boundary of region V to infinity and consider the original problems in the region $R - \Sigma A_n$ (all regions A_n of the infinite periodic lattice enter into the sum). By hypothesis, in R we are given the smooth macrofunctions $v_i(x)$, $u_i(x)$ and $\omega_i(x)$. We assume that they have a finite spectrum (a finite Fourier transform).

Problem 1 reduces to the following auxiliary problem: find the minimum of the functional

$$E_R = 2\mu \int_{R - \Sigma A_n} \varepsilon_{ij} \varepsilon^{ij} d^3x \tag{4.1}$$

under the constraint (1.2) and the additional constraint

$$v_i(n) = \frac{1}{\tau} \int_{B_n - A_n} w_i d^3x, \quad v_i(n) = v_i(x)|_{x=bn} \tag{4.2}$$

The auxiliary problem corresponding to Problem 2 consists in the minimization of functional (4.1) under constraints (4.2), (1.2) and (1.5).

Here $u_i(n) = u_i(x)$ and $\omega_i(n) = \omega_i(x)$ when $x = bn$. The finiteness of the spectra of $v_i(x)$, $u_i(x)$ and $\omega_i(x)$ ensures a sufficiently rapid damping of the solution so that all the integrals over R converge.

We transform formulas (4.1), (4.2), (1.2) and (1.5) to a more convenient form by using the idea of a quasicontinuum [1]

5. **Quasicontinuum.** With each function $G(n^k)$ of integer arguments n^k , defined for all n^k , we can associate a function $G(x^k)$ defined in the whole space R by the rule

$$G(x^k) = \sum_n G(n^k) \delta(x^k - bn^k) \tag{5.1}$$

$$\delta(x^k) = \left(\frac{b}{\pi x^1} \sin \frac{\pi x^1}{b}\right) \left(\frac{b}{\pi x^2} \sin \frac{\pi x^2}{b}\right) \left(\frac{b}{\pi x^3} \sin \frac{\pi x^3}{b}\right)$$

This correspondence possesses the following properties:

$$G(x^k)|_{x^k=bn^k} = G(n^k) \tag{5.2}$$

$$b^3 \sum_n G_1(n^k) G_2(n^k) = \int_R G_1(x^k) G_2(x^k) d^3x \tag{5.3}$$

The interpolating function $G(x^k)$ is "maximally smooth" in x^k relative to a lattice of step b , i. e., its spectrum is concentrated in a cube K with center at zero and with side $2\pi/b$, and, thus, only harmonics with a wavelength greater than b occur in the expansion of $G(x^k)$ into a Fourier integral. Conversely, with each function $G(x^k)$ with a spectrum concentrated in K we can associate a function $G(n^k)$ by formula (5.2); relations (5.1) and (5.3) hold here. The replacement of functions of a discrete argument by functions (5.1) is said to be the introduction of a quasicontinuum.

The finiteness of the spectra of the macrofunctions $v_i(x)$, $u_i(x)$ and $\omega_i(x)$ introduced in Sect. 4 ensures their connection with the functions $v_i(n)$, $u_i(n)$ and $\omega_i(n)$ in constraints (1.5) and (4.2) by formulas (5.1), since for sufficiently small b any finite spectrum falls into K .

6. **Reformulation of the auxiliary problem.** The functions $w_i(x)$ defined in $R - \Sigma A_n$ can be considered as a function of two variables: $w_i = w_i(\xi^k, n^k)$, where ξ^k are cell coordinates and n^k are the components of the integer-valued vector specifying the cell's number. Formulas (4.1), (4.2), (1.2) and (1.5) are rewritten in terms of functions $w_i(\xi^k, n^k)$ as

$$E_R = \sum_n 2\mu b \int_{B-A} w_{(i|j)}(\xi, n) w^{(i|j)}(\xi, n) d^3\xi \tag{6.1}$$

$$v_i(n) = \frac{1}{1-c} \int_{B-A} w_i(\xi, n) d^3\xi$$

$$\frac{\partial w^i(\xi, n)}{\partial \xi^i} = 0 \text{ in } B - A$$

$$w_i(\xi, n) = u_i(n) + e_{ijk} \omega^j(n) b \xi^k, \quad \xi \in \partial A$$

In addition to (6.1) we need to take into consideration that functions w_i are continuous on the cell faces. This condition can be written in terms of $w_i(\xi, n)$ as follows:

$$w_i(\xi^k, n^k)|_{\xi \in \Gamma_s^+} = w_i(\xi^k, n^k + \delta_s^k)|_{\xi \in \Gamma_s^-} \tag{6.2}$$

Here Γ_s^- is the face of cell B with the equation $\xi^s = -1/2$ and Γ_s^+ is the opposite face.

We span the quasicontinuum, i. e., the functions $w_i(\xi, x)$, by the functions $w_i(\xi, n)$. By virtue of property (5.3), the functional takes the form

$$E_R = \int_R \Phi d^3x, \quad \Phi = 2\mu b^{-3} \int_{B-A} w_{(i|j)}(\xi, x) w^{(i|j)}(\xi, x) d^3\xi \tag{6.3}$$

We span the quasicontinuum also by the functions $v_i(n)$, $u_i(n)$ and $\omega_i(n)$. We re-write constraint (6.1) in the form

$$v_i(x) = \frac{1}{1-c} \int_{B-A} w_i(\xi, x) d^3\xi \equiv \langle w_i(\xi, x) \rangle \tag{6.4}$$

$$\frac{\partial w^i(\xi, x)}{\partial \xi^i} = 0, \quad \xi \in B - A, \quad x \in R \tag{6.5}$$

$$w_i(\xi, x) = u_i(x) + b e_{ijk} \omega^j(x) \xi^k, \quad \xi \in \partial A, \quad x \in R \tag{6.6}$$

As is easy to verify, equality (6.2) can be transformed to

$$w_i(\xi^k, x^k)|_{\xi \in \Gamma_s^+} = w_i(\xi^k, x^k + b\delta_s^k)|_{\xi \in \Gamma_s^-} \tag{6.7}$$

In the auxiliary Problem 1 we need to seek the minimum of functional (6.3) over functions $w_i(\xi, x)$ with a spectrum finite in x under constraints (6.4), (6.5) and (6.7); in the auxiliary Problem 2, the minimum of (6.3) under constraints (6.4) - (6.7)

If we put aside the fact that functions w_i have spectra finite in x and are subject to constraints (6.7), then functional (6.3) has the form of the energy functional of Cosserat's generalized continuum: there exists a three-dimensional continuum R , each point of which is provided with its own three-dimensional continuum, viz., the cell $B - A$. Constraint (6.7) connects the cells into a single entity.

7. Problem 1. Functions $v_i(x)$ changes little at distances of the order of b ; therefore, the solution of the problem on the minimum of functional (6.3) under constraints (6.4), (6.5) and (6.7) is naturally sought among the functions $w_i(\xi, x)$ changing little in x at distance of the order of b . In this regard we can replace $w_i(\xi^k, x^k + b\delta_s^k)$ in (6.7) by $w_i(\xi^k, x^k)$ and (6.7) turns into the condition for the periodicity of $w_i(\xi^k, x^k)$ with respect to ξ^k

$$[w_i]_s \equiv w_i(\xi, x)|_{\xi \in \Gamma_s^+} - w_i(\xi, x)|_{\xi \in \Gamma_s^-} = 0 \tag{7.1}$$

The use of the approximate equality (7.1) instead of the exact (6.7) essentially simplifies the investigation, since the problems on the minimum of functional Φ are separate for different x and the possibility appears of minimizing functional (6.3) "point-by-point" by seeking the minimum of Φ for each x . The minimizing function of functional Φ under constraints (6.4), (6.5) and (7.1) obviously has the form $w_i(\xi, x) = v_i(x)$. Since $v_i(x)$ is not determined at the first step, it is necessary to construct the next approximation. We assume $w_i(\xi, x) = v_i(x) + w_i'(\xi, x)$.

Substitution into (6.7) yields $[w_i]_s = bv_{i,s}$. In this connection it is natural to set

$$w_i(\xi, x) = v_i(x) + b\psi_i(\xi, x) \quad (7.2)$$

where the functions $\psi_i(\xi, x)$ satisfy the condition

$$[\psi_i]_s = v_{i,s} \quad (7.3)$$

The substitution of (7.2) into (6.3) leads to a problem on the minimum of functional

$$\Phi = 2\mu \int_{B-A} \psi_{(il)}\psi^{(il)}d^3\xi \quad (7.4)$$

over all ψ_i satisfying condition (7.3) and the incompressibility condition following from (6.5): $\psi_i^i = 0$. After the change of variables $\psi_i \rightarrow \psi_i + \xi^s v_{i,s}$ the problem on the minimum of functional (7.4) turns into a problem on the minimum of functional (3.3).

The minimal value U of the functional Φ is a quadratic form in $v_{i,s}$:

$U = 1/2 E^{ijkl} v_{i,j} v_{k,l}$. In the first approximation, for the minimal value \bar{E}_R of the functional E_R in (6.3) we can write

$$\bar{E}_R = \int_R U d^3x$$

It is obvious that the substitution of (7.2) into (1.6) yields, to within small quantities of higher order in b ,

$$\bar{E} = \int_V U d^3x \quad (7.5)$$

and the equations for determining $v_i(x)$ are the Euler equations of functional (7.5).

8. Problem 2. As in Problem 1 we seek the solution among the functions $w_i(\xi, x)$ changing little in x at distances of the order of b , and, in this connection we replace constraints (6.7) by the approximate equalities (7.1). Once again we can solve the problem on the minimum of functional (6.3) point-by-point, seeking for each x the minimum of functional Φ under constraints (7.1) and (6.4)-(6.6). The essential difference from Problem 1 is that the minimizing functions w_i^* depend on ξ even at the first step and the minimal value of functional Φ is nonzero. The minimizing functions w_i^* can by virtue of the problem's linearity be represented in the form

$$w_i^* = v_i(x) + (u_k(x) - v_k(x)) \varphi_i^k(\xi), \quad \langle \varphi_i^k \rangle = 0. \quad (8.1)$$

For brevity, together with φ_i^k we shall operate with the functions $\varphi_i = (u_k - v_k) \varphi_i^k$, keeping in mind that the subsequent substitution of the relation for φ_i and the use of the arbitrariness of $u_k - v_k$ yield the equations for φ_i^k . The constraints on w_i take the form

$$\partial \varphi_i^i / \partial \xi^i = 0, \quad \langle \varphi_i \rangle = 0, \quad [\varphi_i]_s = 0, \quad \varphi_i |_{\partial A} = u_i - v_i \quad (8.2)$$

From the condition of the minimum of functional Φ we find the equations for φ_i

$$-\partial p / \partial \xi^i + 2\mu \Delta \varphi_i + \lambda_i = 0, \quad [-p \delta_{ij} + 2\mu \varphi_{(il)j}]_j = 0 \quad (8.3)$$

In (8.3) there is no summation on j ; p and λ_i are the Lagrange multipliers for the first two constraints in (8.2).

The minimal value D_1 of functional Φ is a quadratic form in the relative velocity

$$D_1 = D^{ij} (u_i - v_i) (u_j - v_j) \tag{8.4}$$

Let us consider the following approximation with respect to b . Substitution of (8.1) into (6.7) shows that the discrepancy is of order b . Therefore, we seek w_i in the form

$$w_i = w_i^* + b\psi_i \tag{8.5}$$

where $\psi_i(\xi, x)$ changes little in x at distances of the order of b . Substitution of (8.5) in (6.7) with the use of (8.1) and of the periodicity condition for φ_i^k yields

$$[\psi_i]_s = v_{i,s} + \Delta_{ks} \varphi_i^k, \quad \Delta_{ks} = u_{k,s} - v_{k,s} \tag{8.6}$$

The functions ψ_i satisfy, besides (8.6), the constraints

$$\partial\psi_i/\partial\xi^i = 0, \quad \psi_i|_{\partial A} = e_{ijk} \omega^j(x) \xi^k \tag{8.7}$$

From (8.6) and (8.7) it follows that the quantities $v_{i,s}$ cannot be arbitrary. As a matter of fact, by integrating the first equality in (8.7) over $B - A$, we obtain

$$\sum_{s=1}^3 \int_{\Gamma_s^+} [\psi_s]_s d^2\xi + \int_{\partial A} e_{ijk} \omega^j \xi^k v^i d^2\xi = 0 \tag{8.8}$$

The last integral in (8.8) equals zero. Using (8.6), for the summation in (8.8) we can write

$$\begin{aligned} \sum_{s=1}^3 \int_{\Gamma_s^+} [\psi_s]_s d^2\xi &= \sum_{s=1}^3 \int_{\Gamma_s^+} (v_{s,s} + \Delta_{ks} \varphi_s^k) d^2\xi = \\ v_{,i}^i + \Delta_{ks} \int_{\partial B} \varphi_j^k \xi^s v^j d^2\xi &= \\ v_{,i}^i - \Delta_{ks} \int_{\partial A} \varphi_j^k \xi^s v^j d^2\xi + \Delta_{ks} \int_{\partial(B-A)} \varphi_j^k \xi^s v^j d^2\xi \end{aligned} \tag{8.9}$$

The last term in (8.9) equals zero (to be convinced of this it is sufficient to pass to an integration over $B - A$ and to use equality (8.2)). By virtue of the boundary condition $\varphi_i^k = \delta_i^k$ on ∂A the integral in the second term equals $-c\delta^{ks}$. Therefore, as the condition for the solvability of the equations for ψ_i we obtain from (8.8) and (8.9) the equation of continuity for the continuous and discrete phases

$$\partial/\partial x^i ((1 - c) v^i + cu^i) = 0 \tag{8.10}$$

The substitution of (8.5) into functional Φ leads to the following expression:

$$\begin{aligned} \Phi &= D_1 + D_{12} + 2\mu \int_{B-A} \psi_{(i|j)} \psi^{(i|j)} d^3\xi \\ D_{12} &= 4\mu b^{-1} \int_{B-A} \varphi_{(i|j)} \varphi^{(i|j)} d^3\xi \end{aligned}$$

With the use of an integration by parts and of Eqs. (8.3), (8.6) and (8.7) we can convince ourselves that D_{12} depends only on φ_i and on the parameters occurring in the boundary conditions for ψ_i . Therefore, the minimization of Φ over ψ_i

is reduced to the second variational problem on a cell, described in Sect. 3.

The minimal value D of functional Φ is the sum $D = D_1 + D_{12} + D_2$ and the averaged equations are the Euler equations of the functional

$$\bar{E} = \int_V D d^3x$$

under constraint (8.10) and condition $v_i = h_i$ on ∂V .

9. On the values of the coefficients in Problem 2 in the case of small concentrations. As $c \rightarrow 0$, D_1 passes into dissipation (referred to the cell volume) which is caused by the translational motion of body A in an unbounded viscous liquid. In particular, for a sphere of radius a [19]

$$D_1 = 6 \pi \mu a b^{-3} (u - v)^2$$

If the body has cubic symmetry, it follows from the properties of tensor-valued functions that $D_{12} = 0$.

Let us compute D_2 when $\Delta_{ij} = 0$. We represent ψ_i as the sum $\psi_i = \psi_i' + v_{i,j} \xi^j$. Then D_2 is the minimal of a functional

$$D_2 = \inf 2\mu \int_{B-A} (\psi'_{(ij)} + e_{ij})(\psi'^{(ij)} + e^{ij}) d^3\xi \quad (9.1)$$

The minimum of functional (9.1) is taken over all functions ψ_i' satisfying, according to (8.6) and (8.7), the conditions

$$\partial \psi'^i / \partial \xi^i = 0, \quad [\psi_i']_s = 0, \quad \psi_i' |_{\partial A} = -e_{ij} \xi^j + e_{ijk} \Omega^j \xi^k \quad (9.2)$$

The first equality in (9.2) has been obtained with due regard to the fact that by virtue of (8.10) $v_{,i}^i = 0$ when $\Delta_{ij} = 0$. Removing the parentheses in (9.1) we have

$$D_2 = \inf \left[2\mu (1 - c) e_{ij} e^{ij} + 2\mu e_{ij} 2 \int_{B-A} \psi'^{(ij)} d^3\xi + 2\mu \int_{B-A} \psi'_{(ij)} \psi'^{(ij)} d^3\xi \right] \quad (9.3)$$

The second summand in (9.3), after an integration by parts using (9.2), equals $4 \mu c e_{ij} e^{ij}$. Therefore,

$$D_2 = 2 \mu (1 + c) e_{ij} e^{ij} + D_2' \quad (9.4)$$

The quantity D_2' is the minimal value of the last summand in (9.3) over all functions ψ_i' satisfying constraints (9.2). Formula (9.4) is valid for body A of arbitrary form for any value of concentration c . As $c \rightarrow 0$, D_2' passes into dissipation (referred to the cell's volume) which is caused in the liquid's infinite volume by the homogeneous deformation of body A with deformation velocity tensor e_{ij} and by the rotation of A with angular velocity Ω_i . In the case of a sphere we can calculate that

$$D_2 = 3 \mu c (e_{ij} e^{ij} + 2 \Omega_i \Omega^i). \quad \text{Thus}$$

$$D_2 = 2 \mu (1 + 5/2 c) e_{ij} e^{ij} + 6 \mu c \Omega_i \Omega^i \quad (9.5)$$

Expression (9.5) shows that the correction to the shear viscosity is indeed given by the Einstein formula.

The well-known derivations of the Einstein correction do not differ in rigor and have repeatedly evoked discussions. In addition, from (9.5) we see that the dissipation contains the curl of the velocity. As a result of this the viscous stress tension $\frac{1}{2} \partial D / \partial v_{i,j}$ which occurs in the averaged equations (3.7) is asymmetric. Mutually exclusive assertions also have been made regarding the symmetry of the stress tensor. We remark that because of the smallness of the rotational viscosity coefficient the situations in which the skew-symmetric part of the stresses make an essential contribution turn out to be rather special. According to (9.5) the coefficients of the averaged equations, in the limit as $c \rightarrow 0$, are independent of the lattice parameters. In this is manifested a more general property of the systems being examined; for small concentrations the interactions of the particles can be neglected while the efficiency coefficients depend only on the form of the particles but not on their disposition. The proof of this statement has been given in [20]

10. On generalizations. What has been presented can be immediately extended to any linear equations which are the Euler equations of a convex quadratic functional. The generalization to Euler equations of a convex nonquadratic functional when Dirichlet-type conditions are set on the cavities, is carried out by averaging the original functional on fields of form (3.5), analogously to the way this is done in [5] for the Neumann-type conditions.

Generally speaking, Assumptions 1 and 2 in Sect. 2 are incorrect in dynamic problems. Therefore, here we merely describe briefly a method for constructing certain particular solutions which in a specific sense change little at distances of the order of b . We restrict ourselves to problems on the oscillations of an infinite elastic body containing periodically situated traction-free cavities. The exact solutions are the extremals of the functional

$$I = \int_{t_1}^{t_2} \left[\int_{R - \Sigma A_n} (\lambda (\varepsilon_i^i)^2 + 2\mu \varepsilon_{ij} \varepsilon^{ij} - \rho w_{i,t} w_{i,t}^i) d^3x \right] dt \quad (10.1)$$

After passing to the quasicontinuum, functional (10.1) takes the form

$$I = \int_{t_1}^{t_2} \int_R \Phi d^3x dt \quad (10.2)$$

$$\Phi = \int_{B-A} [b^{-2} (\lambda (w_{i,t}^i)^2 + 2\mu w_{(i|j)} w^{(i|j)}) - \rho w_{i,t} w_{i,t}^i] d^3\xi$$

Assuming that the functions w_i change slowly in x at distance b , we pass from the exact joining conditions (6.7) to the approximate (7.1). Then the search for the extremals of functional I is reduced to the search for the extremals of functional Φ . The latter have the form

$$w_k = F_k(\xi) e^{i\omega t}$$

Here $F_k(\xi)$ and ω are the eigenfunctions and eigenvalues of the problem on the natural oscillations of cell with periodic boundary conditions

$$\sigma_{ij}^{ij} + \omega^2 F^i = 0, \quad \sigma_{ij} = \lambda F_{|k}^k \delta_{ij} + 2\mu F_{(i|j)}, \quad [F^i]_s = 0, \quad [\sigma^{ij}]_s v_j = 0$$

Accountable number of natural oscillations exists. To each of them corresponds its own slowly-varying solution. Let us first consider the natural oscillations corresponding to zero natural frequency ω : the eigenfunctions F_i are independent of ξ ; therefore, $w_i = u_i(x)$. We see that we can adopt a more general dependency

$$w_i = u_i(x, t) \quad (10.3)$$

It is sufficient only that $\partial u_i / \partial t \sim \sqrt{\mu/\rho} \partial u_i / \partial x$. Similar oscillations can be called slow or quasistatic. The substitution of (10.3) into the exact joining conditions (6.7) shows that

$$w_i = u_i(x, t) + b \psi_i(\xi, x, t), \quad [\psi_i]_s = u_{i,s}$$

For the slow oscillations we can omit the dynamic terms in the equations for ψ_i . The averaged equations will differ from the averaged static equations by the summand $\langle \rho \rangle \partial^2 u_i / \partial t^2$.

Let us consider the natural oscillations with nonzero frequency ω . The eigenfunctions are determined to within a multiplier u not depending on ξ : $w_k = u(x) F_k(\xi) e^{i\omega t}$. We can adopt a more general formula for the first term of the asymptotics: $w_k = u(x, t) \cdot F_k(\xi)$. Here it is assumed that $\partial u / \partial t \sim \omega u$. Any nonzero natural frequency ω is of the order $b^{-1} \sqrt{\mu/\rho}$. Therefore, the corresponding "macro-oscillations" defined by functions $u(x, t)$ are "rapid". The next term of the asymptotics and the averaged equations are found by the same scheme as in the static case. The averaged equations have the form of the Klein - Gordon equations $A \Delta u - B u = u_{tt}$.

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